Topological aspects of Cardy-Rabinovici model

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Strings and QFTs for Eurasian time zone

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Motivation: Yang-Mills dynamics and the θ angle

Yang-Mills theory

We consider $SU(N)$ Yang-Mills theory:

$$\mathcal{L} = \frac{1}{g^2}|F|^2.$$

YM coupling g^2 is not a free parameter, but defines the strong scale:

$$\Lambda \equiv \mu \exp\left(-\frac{8\pi^2}{\left(\frac{11}{3}N - \frac{2}{3}n_f\right) g^2(\mu)}\right) .$$

Instantons and θ angle

4d Yang-Mills theory also has the nonperturbative parameter: θ angle

$$\mathcal{L} = \frac{1}{g^2}|F|^2 + \mathrm{i}\,\theta\, F\tilde{F}.$$

This term is a total derivative, so it does not appear in EOM:

$$D_\mu F_{\mu\rho} = 0.$$

But, this total derivative term counts the number of instantons:

$$\int_{M_4} F\tilde{F} = n \in \mathbb{Z}.$$

(Belavin, Polyakov, Schwarz, Tyupkin, '75; Callan, Dashen, Gross, '76)

Large-N confined phase of $SU(N)$ YM

't Hooft large-N limit:

$$N \to \infty \quad \text{with} \quad \lambda = g^2 N \quad \text{fixed.}$$

We can rewrite the path integral as

$$Z(\theta) = \int \mathcal{D}a \exp\left[-N \int \left(\frac{1}{2\lambda}|F|^2 + \mathrm{i}\frac{\theta/N}{8\pi^2}F^2\right)\right].$$

In the $N \to \infty$ limit, θ/N is more natural variable than θ itself.
(Witten, '80, '98)

Multi-branch structure of confined vacua

Since θ/N is quite small as $N \to \infty$,

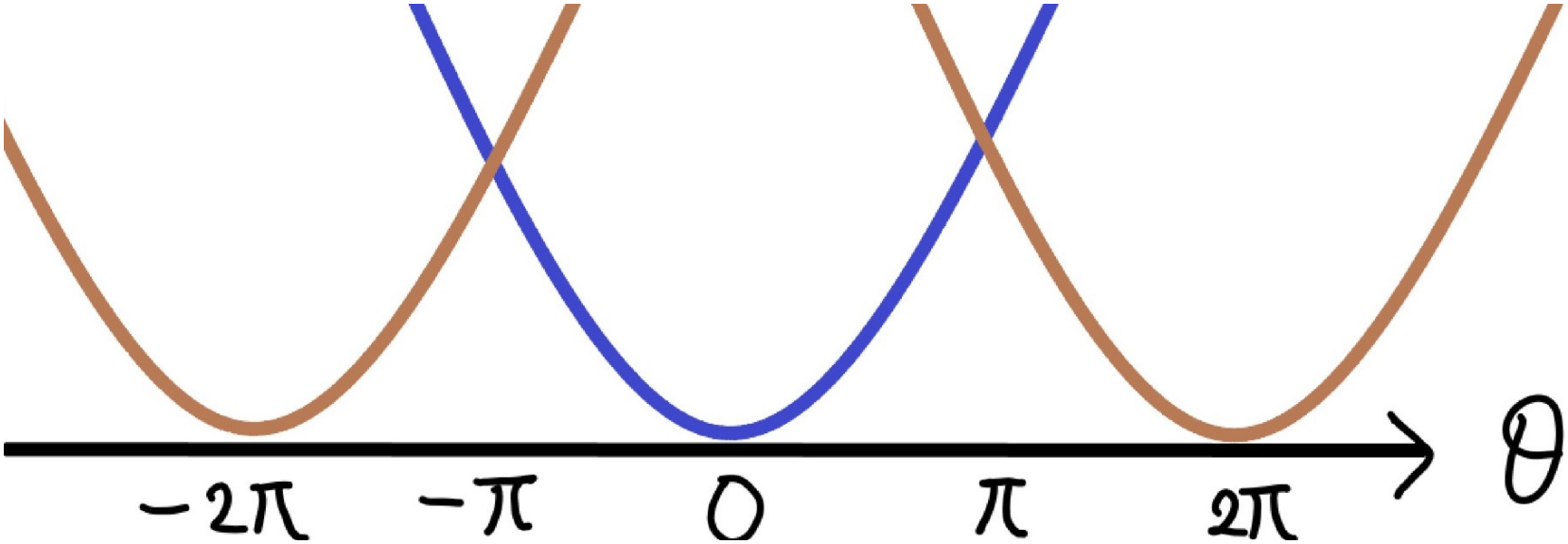
$$\begin{aligned} \mathcal{E}(\theta) &= N^2 f(\theta/N) \\ &= N^2 f(0) + \frac{f''(0)}{2}\theta^2 + O(\theta^4/N^2). \end{aligned}$$

$\chi_{\rm top} = f''(0)$ is the topological susceptibility ($\sim \Lambda^4$).

However, $\mathcal{E}(\theta)$ must be 2π periodic, while the above is not.
$\Rightarrow$ Multi-branch structure (Witten, '80, '98):

$$\mathcal{E}(\theta) = \min_{k\in\mathbb{Z}} \frac{\chi_{\rm top}}{2}(\theta - 2\pi k)^2.$$

Schematic illustration of $\mathcal{E}(\theta)$:



At $\theta = \pi$, the level crossing occurs: 1st order phase transition.
CP symmetry at $\theta = \pi$ is spontaneously broken.

Model: Cardy-Rabinovici model

Model of dual superconductor

Nambu, Mandelstam, 't Hooft :

$$\text{Confinement} = \text{condensation of monopoles}$$

In the Lagrangian formulation, however, we do not know how to write down electric and magnetic excitations simultaneously.

Instead, Cardy and Rabinovici constructed such a lattice model including a θ term. (Cardy, Rabinovici, '82)

Cardy-Rabinovici model (continuum ver.)

Cardy-Rabinovici model is a $U(1)$ gauge theory with θ:

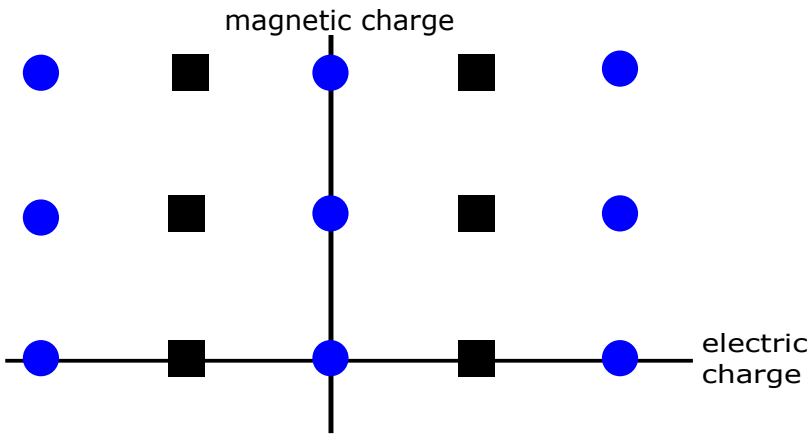
$$Z = \int \mathcal{D}a \, \exp\left(-\frac{1}{2g^2} \int |\mathrm{d}a|^2 + \mathrm{i}\frac{N\theta}{8\pi^2} \int \mathrm{d}a \wedge \mathrm{d}a \right) Z_{\mathrm{matter}}[a]$$

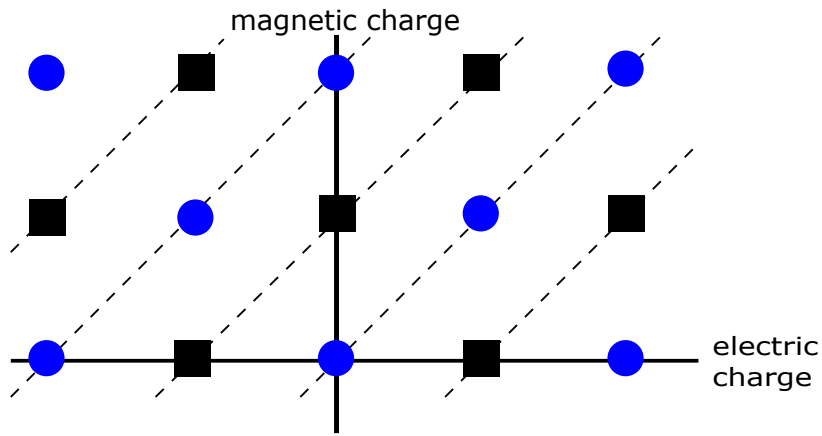
It includes both electric matter of charge N (Wilson loops, $W^N(\ell)$) and magnetic matter ('t Hooft loops $H(\ell)$):

$$Z_{\mathrm{matter}}[a] = \sum_{\{n_\mu\},\{m_\mu\}} \exp(-S_{\mathrm{matter}}[\{n_\mu\},\{m_\mu\}])\, W^N(\{n_\mu\}) H(\{m_\mu\})$$

Role of θ angle: Witten effect

At finite θ, magnetic charge acquires the electric charge:





That is, the particle of type (n, m) has the ElectroMagnetic charge

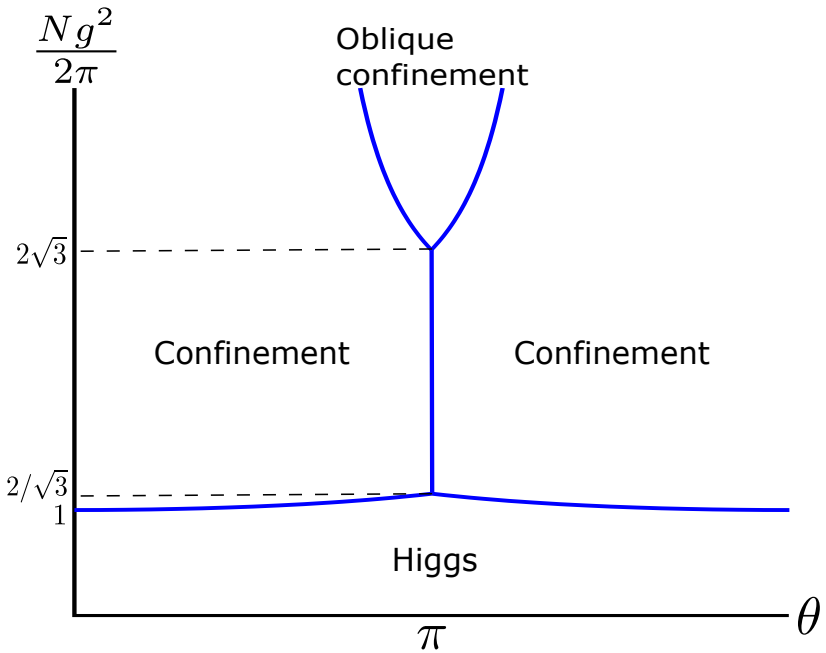
$$(E, M) = \left(N(n + \frac{\theta}{2\pi} m), m \right) .$$

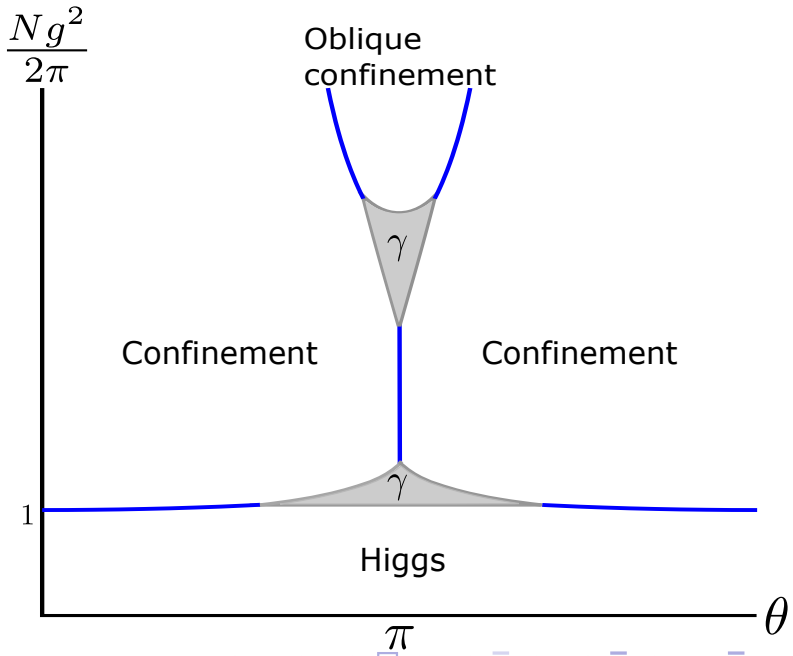
Rich phase diagram

Condensation of the particle type (n, m) would produce the free-energy

$$F_{n,m} \sim g^2 E^2 + \left(\frac{2\pi}{g}\right)^2 M^2 = g^2 N^2 \left(n + \left(\frac{\theta}{2\pi}\right) m\right)^2 + \left(\frac{2\pi}{g}\right)^2 m^2.$$

Depending on g and θ, we should pickup (n, m) giving the minimal free-energy. (Cardy, Rabinovici, '82)





Modular invariance of the phase diagram

Introducing the complex coupling,

$$\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{2\pi}{Ng^2},$$

the free energy $F_{n,m}$ takes a nice expression:

$$F_{n,m} \sim \frac{1}{\mathrm{Im}(\tau)}|n + m\tau|^2.$$

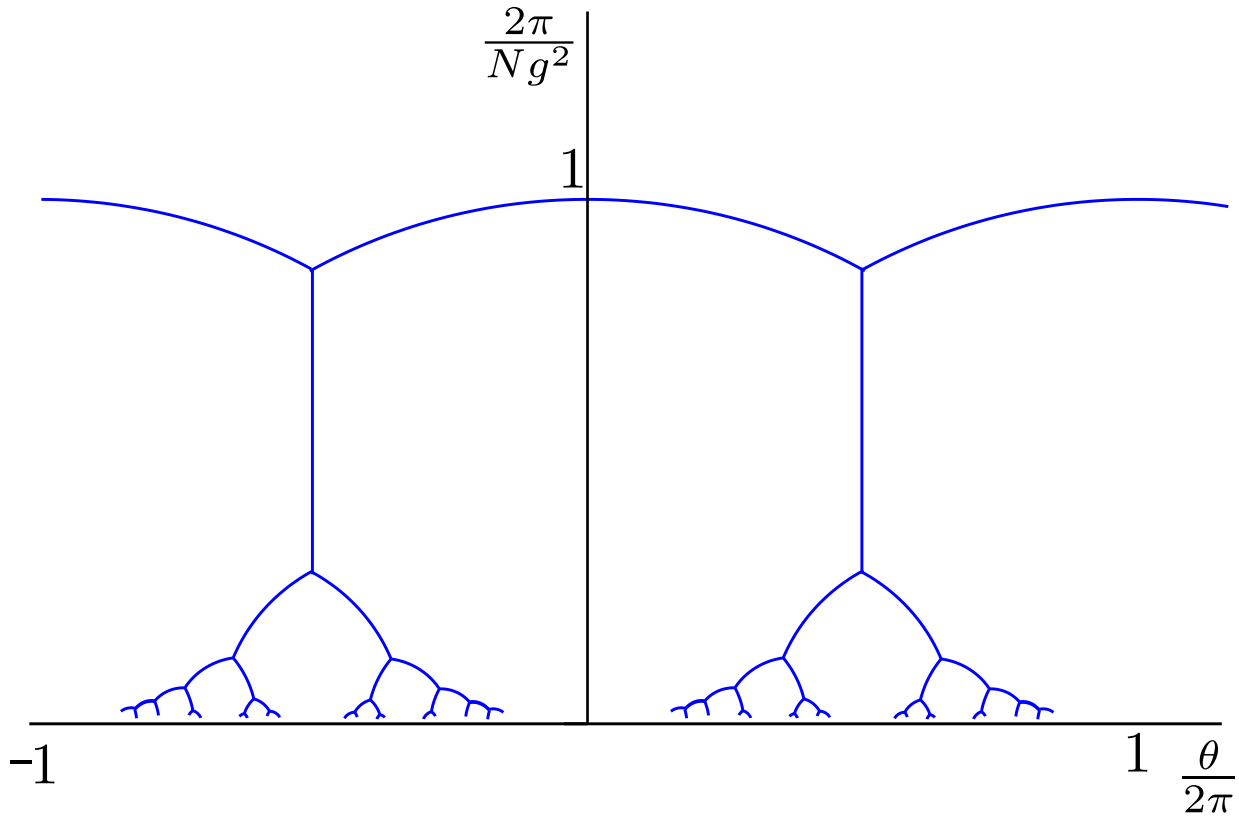
This suggests that the local dynamics enjoys the $SL(2,\mathbb{Z})$ duality

(Cardy '82)

$$S : \tau \to -\frac{1}{\tau},\ (n,m) \to (-m,n).$$

$$T : \tau \to \tau + 1,\ (n,m) \to (n-m,m).$$

Very Rich phase diagram



Not only familiar phases, Higgs and confinement, but also many new phases, oblique confinement, will appear!! (Cardy, Rabinovici, '82, Cardy, '82)

Anomaly matching: Topological aspects of Cardy-Rabinovici model

Anomaly matching

Let's assume we have a QFT with global symmetry G

Definition ('t Hooft anomaly)

Introduce Background gauge field A for G.

$$Z[A+\delta_\theta A] = \exp\left(\mathrm{i}\mathcal{A}[A,\theta]\right) Z[A].$$

Anomalous phase $\mathcal{A}[A,\theta]$ is nowadays called an 't Hooft anomaly.

Theorem ('t Hooft anomaly matching)

't Hooft anomaly is renormalization-group invariant. *('t Hooft, '79, '80)*
Low-energy effective theory must reproduce the same $\mathcal{A}[A,\theta]$.

Symmetry of Cardy-Rabinovici model

Since the electric excitations must have the charge N, the theory has the 1-form symmetry:

$$\mathbb{Z}_N^{[1]} : W(C) \to \mathrm{e}^{2\pi \mathrm{i}/N} W(C).$$

Another important symmetry is the CP symmetry. Since

$$\mathrm{CP} : \theta \to -\theta,$$

CP is not a symmetry for generic θ. Since $\theta \sim \theta + 2\pi$, it is a symmetry only at

$$\theta = 0, \quad \pi.$$

Mixed anomaly between $\mathbb{Z}_N^{[1]}$ and CP at $\theta = \pi$

Let us introduce a background 2-form gauge field B couple to $\mathbb{Z}_N^{[1]}$,

$$Z_\theta[B].$$

With this background, the θ periodicity is no longer 2π:

$$Z_\theta[B] \xrightarrow{\theta\to\theta+2\pi} Z_{\theta+2\pi}[B] = \exp\left(\frac{\mathrm{i}N}{4\pi}\int B\wedge B\right) Z_\theta[B].$$

This gives the mixed anomaly at $\theta = \pi$ (for even N),

$$\mathrm{CP}: Z_{\theta=\pi}[B] \to \exp\left(-\frac{\mathrm{i}N}{4\pi}\int B\wedge B\right) Z_{\theta=\pi}[B].$$

Note: Same anomaly exists for pure YM (Gaiotto, Kapustin, Komargodski, Seiberg, '17)

Anomaly matching in Higgs/confinement phases

Anomaly matching suggests that $\mathbb{Z}_N^{[1]} \times$ CP should be spontaneously broken down to "an anomaly-free subgroup".

Higgs In the Higgs phase, all the Wilson loops obey the perimeter law, showing that

$$\mathbb{Z}_N^{[1]} \to 1.$$

Confinement In the confinement phase, CP is broken at $\theta = \pi$:

$$(\mathbb{Z}_2)_{\mathsf{CP}} \to 1.$$

This suggests that monopole condensation $(n, m) = (0, 1)$ and dyon condensation $(n, m) = (-1, 1)$ are different as SPT:

$$Z_{\mathrm{mon}}[B] \sim 1, \quad Z_{\mathrm{dyon}}[B] \sim \exp\left(\frac{\mathrm{i}N}{4\pi}\int B^2\right).$$

Oblique confinement

Oblique confinement at $\theta = \pi$ is caused by the condensation of $(n, m) = (-1, 2)$.

Why such an exotic condensation is preferred??

$$F_{n,m} \sim g^2 N^2 \underbrace{\left(n + \left(\frac{\theta}{2\pi} \right) m \right)^2}_{=0 \text{ if } (n,m)=(-1,2),\, \theta=\pi} + \left(\frac{2\pi}{g} \right)^2 m^2$$

When the electric coupling g^2 is sufficiently large, we should make the net electric charge E as small as possible, even if we must cost M.

Oblique confinement and $\mathbb{Z}_2$ topological order

The loop operator obeys perimeter law only if its charge is mutually local with that of condensation, $(n, m) = (-1, 2)$, in mod N.

For even N,

- $W^{N/2}$ obeys perimeter law.
- W^k $(k \neq N/2)$ obeys the area law.

This suggests the symmetry breaking,

$$\mathbb{Z}_N^{[1]} \to \mathbb{Z}_{N/2}^{[1]}.$$

Let us check if the unbroken symmetry is anomaly-free.

Anomaly matching for oblique confinement phase

In order to see if $\mathbb{Z}_{N/2}^{[1]} \times$ CP is anomaly free (for even N), we replace B to the $\mathbb{Z}_{N/2}$ 2-form gauge field B':

$$\text{CP} : Z_\pi[B'] \to \exp\left(-\mathrm{i}\frac{\frac{N}{2}\cdot 2}{4\pi}\int B'^2\right) Z_\pi[B'].$$

At first sight, it looks to be still anomalous for $N \geq 4$.
We can however add the local counterterm of B':

$$\tilde{Z}_\pi[B'] = Z_\pi[B'] \exp\left(-\mathrm{i}\frac{\frac{N}{2}}{4\pi}\int B'^2\right),$$

which is CP invariant at $\theta = \pi$.

Oblique confinement gives a minimal breaking scenario of $\mathbb{Z}_N^{[1]}$ to match the anomaly!!

$SL(2,\mathbb{Z})$ duality: Gauged Cardy-Rabinovici model

Duality of local dynamics $\not\Rightarrow$ Global self-duality

Free-energy argument suggests the Cardy-Rabinovici model enjoys

$$S : \tau \to -\frac{1}{\tau}, (n,m) \to (-m,n), \quad T : \tau \to \tau+1, (n,m) \to (n-m,m).$$

This is indeed true for local dynamics.

Globally, there is a problem. S is an electromagnetic duality. Cardy-Rabinovici model, however, only has the electric 1-form symmetry, $(\mathbb{Z}_N^1)_E$.
$\Rightarrow$ S transformation maps one theory to a different theory with the identical local dynamics.

Gauged Cardy-Rabinovici model

Let us construct a $SL(2,\mathbb{Z})$ self-dual theory out of the Cardy-Rabinovici model.

If the electric and magnetic 1-form symmetry takes the form,

$$(\mathbb{Z}_M^{[1]})_E \times (\mathbb{Z}_M^{[1]})_M,$$

then the previous issue no longer occurs.

We can obtain such a theory for $N = M^2$ and gauge the subgroup of 1-form symmetry,

$$\mathbb{Z}_M^{[1]} \subset \mathbb{Z}_{N=M^2}^{[1]}.$$

(We can check explicitly that this theory enjoys the $SL(2,\mathbb{Z})$ self-duality.)

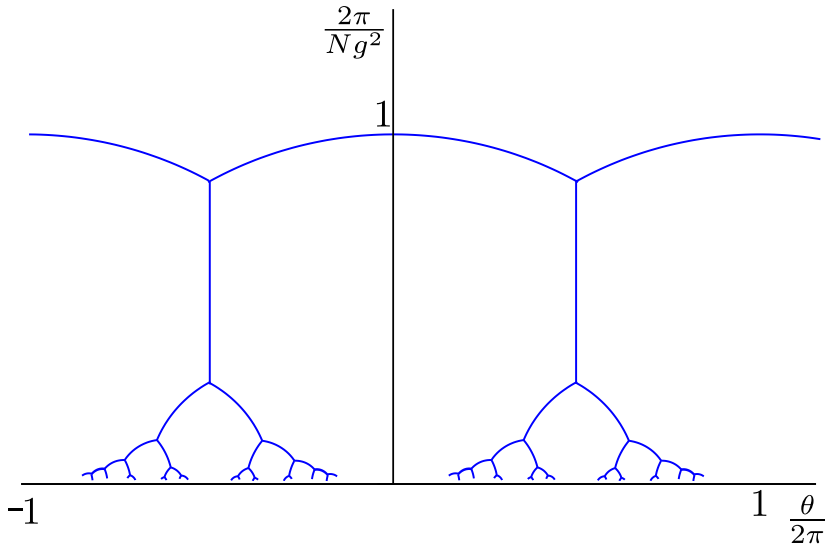
Enhancement of global symmetry due to $SL(2,\mathbb{Z})$

Now, we have a theory with $SL(2,\mathbb{Z})$ self-duality.

$\Rightarrow$ If τ is a nontrivial fixed point, its stabilizer subgroup is an extra symmetry.

Examples:

- $\tau = \mathrm{i}$ is a fixed point of S: $S : \tau = \mathrm{i} \to -1/\tau = \mathrm{i} = \tau$. Thus, there is a $\mathbb{Z}_4$ symmetry generated by S.
- $\tau = \mathrm{e}^{\pi \mathrm{i}/3}$ is a fixed point of ST^{-1}, i.e. $(\mathbb{Z}_6)_{ST^{-1}}$ symmetry.



Mixed $(\mathbb{Z}_6)_{ST^{-1}}$-gravity anomaly

It has been known that the Maxwell partition function is a modular form, instead of being modular invariant (Witten, '95, Verlinde, '95):

$$\begin{aligned} Z_{\text{Maxwell}}(\tau+1) &= Z_{\text{Maxwell}}(\tau), \\ Z_{\text{Maxwell}}(-1/\tau) &= \tau^{\frac{1}{4}(\chi-\sigma)}\overline{\tau}^{\frac{1}{4}(\chi+\sigma)} Z_{\text{Maxwell}}(\tau). \end{aligned}$$

Here, χ is the Euler number of the spin four-manifolds, and σ is the signature of the spin four-manifolds,

$$\sigma = \frac{1}{3}\int \frac{1}{8\pi^2}\mathrm{tr}(R\wedge R).$$

Anomaly matching of $(\mathbb{Z}_6)_{ST^{-1}}$-gravity anomaly

We then find the mixed anomaly at $\tau = \mathrm{e}^{\pi \mathrm{i}/3}$ (Seiberg, Tachikawa, Yonekura, '18)

$$(\mathbb{Z}_6)_{ST^{-1}} : Z(\tau = \mathrm{e}^{\pi \mathrm{i}/3}) \to \exp(2\pi \mathrm{i} \frac{\sigma}{6}) Z(\tau = \mathrm{e}^{\pi \mathrm{i}/3}).$$

On the spin manifold $K3$, $\sigma = -16$, and it is indeed a nontrivial anomaly.

Anomaly matching suggests the symmetry breaking,

$$(\mathbb{Z}_6)_{ST^{-1}} \to \mathbb{Z}_2.$$

This explains the triple degeneracy of the conjectured phase diagram at $\tau = \mathrm{e}^{\pi \mathrm{i}/3}$.

Summary

- We revisit the Cardy-Rabinovici model from the topological viewpoints.
- Cardy-Rabinovici model gives an intuitive explanation of various possible phases of non-Abelian gauge theories.
- We identify how the oblique confinement phase can match the mixed anomaly at $\theta = \pi$. $\mathbb{Z}_N^{[1]} \to \mathbb{Z}_{N/2}^{[1]}$ is a minimal scenario for anomaly matching.
- We construct the $SL(2,\mathbb{Z})$ self-dual theory out of the Cardy-Rabinovici model. It has an interesting mixed gravitational anomaly, giving a constraint on the phase diagram.